Discrete rotations in Mojette space

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Outline

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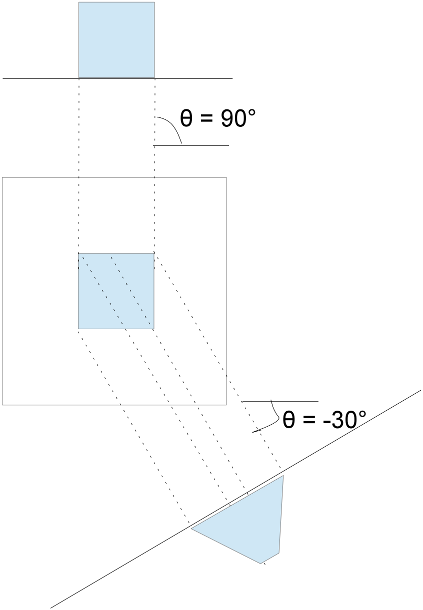
Introduction

Radon transform

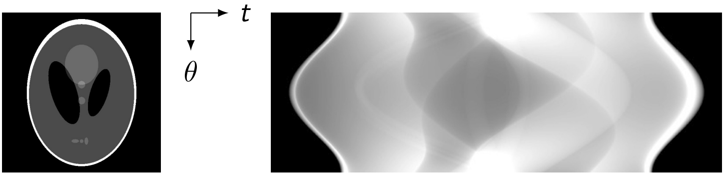
The Radon transform (1917) represents a function f by its integrals along straight lines (projections) :

$$[Rf](t,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(t - x\cos\theta - y\sin\theta)dxdy$$

Radon transform



Sinogram and tomography



(a) (b)

FIGURE: (a) Shepp-Logan phantom. (b) Sinogram

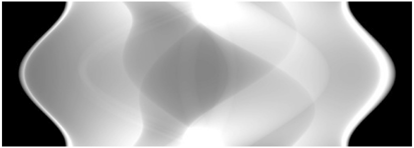
Sinogram Set of projections of an object by the Radon transform. Each line represents a projection

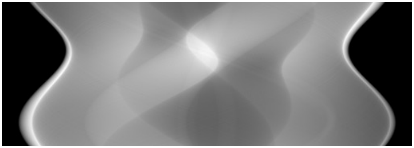
Tomographic reconstruction Inverse problem of object recovery from the sinogram.

Sinogram rotation

In Radon sinogram space

- The space is cyclic in angular direction with π-periodicity
- A θ angle rotation corresponds to an offset of the first projection (cyclic translation)





Mojette transform (GUEDON, BARBA et BURGER, 1995)

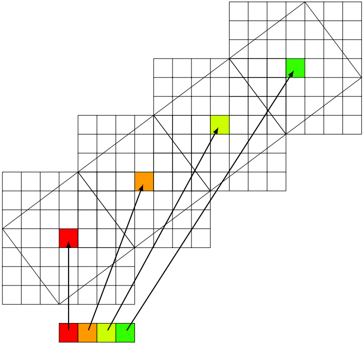
- Discrete image $f(k,l)$
- Discrete directions (p,q) with p and q coprime integers
- The projection values (*bins*) correspond to the sum of the pixels sampled by the discrete ray

$$[\mathrm{MT}f](b,p,q) = \sum_{k=0}^{P-1} \sum_{l=0}^{Q-1} f(k,l)\Delta(b + kq - lp)$$

Rotations in Mojette projection space

Rotation in Mojette projection space

Inspired by FRT space transforms (SVALBE, 2011), let us define a rotation in Mojette space. Every projection is transformed into a new projection with a new direction. Revertilibity is ensured by the upscale.



Connex pixels of the image are not connected (in the common sense) anymore after a (p, q) rotation, but are aligned on a (p, q)-directed line.

Mojette rotation : algorithm

Algorithm for the upscaled Mojette rotation :

For each projection P of angle (p, q).

1. Generate the new projection P' of angle (p', q')
2. 1D convolution to fill "empty" space

Then, reconstruct the image from new projections.

Computation of the new projection direction

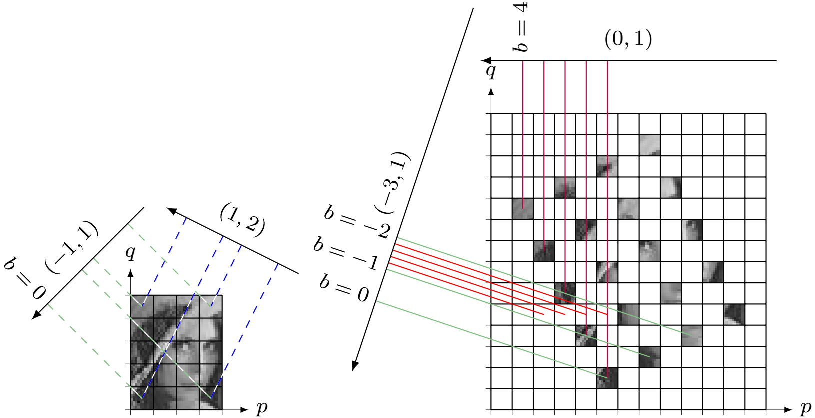
- (p, q) rotation matrix : $R_{(p_\theta, q_\theta)} = \begin{pmatrix} p_\theta & -q_\theta \\ q_\theta & p_\theta \end{pmatrix}$
- New projection direction (p', q') :

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} p_\theta & -q_\theta \\ q_\theta & p_\theta \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} pp_\theta - qq_\theta \\ pq_\theta + qp_\theta \end{pmatrix}$$

- To ensure p' and q' are coprime, we normalize them by their greatest common divisor :

$$\begin{pmatrix} p'' \\ q'' \end{pmatrix} = \frac{1}{\gcd(p', q')} \begin{pmatrix} p' \\ q' \end{pmatrix}$$

New projection computation



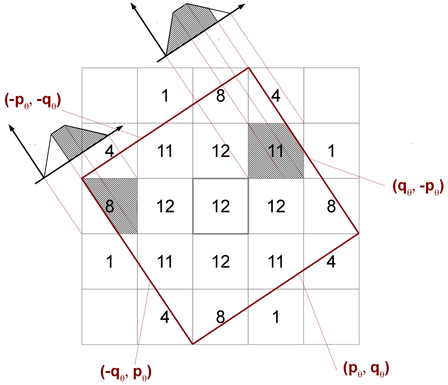
New projection computation

- Bins values don't change during the rotation process because the same pixels are sampled (or sometimes only null valued additionnal pixels).
- We get the new projection by oversampling the initial projection with factor :

$$S = \frac{p_\theta^2 + q_\theta^2}{\text{pgdc}(p', q')}$$

Filling the new projections

- Oversampling → A lot of null pixels
- Interpolation by weighting each pixel by the overlapping area ratio between a "meta" pixel corresponding to the initial pixel rotation and the initial grid



Convolution kernel computation

Problem : how to compute efficiently the overlapping area between black pixels and red "meta-pixel"

- Meta-pixel size : $|p_\theta| + |q_\theta|$
- For each pixel (k, l) of the convolution kernel, we compute the distance between its center and the closest edge (red colored on the figure) :

$$d_{p_\theta,q_\theta}(k,l) = \left\| \begin{pmatrix} -q_\theta & p_\theta \\ p_\theta & q_\theta \end{pmatrix} \cdot \begin{pmatrix} k \\ l \end{pmatrix} \right\|_\infty \quad (1)$$

$$x_{p_\theta,q_\theta}(k,l) = \frac{p_\theta^2 + q_\theta^2}{2} - d_{p_\theta,q_\theta}(k,l) \quad (2)$$

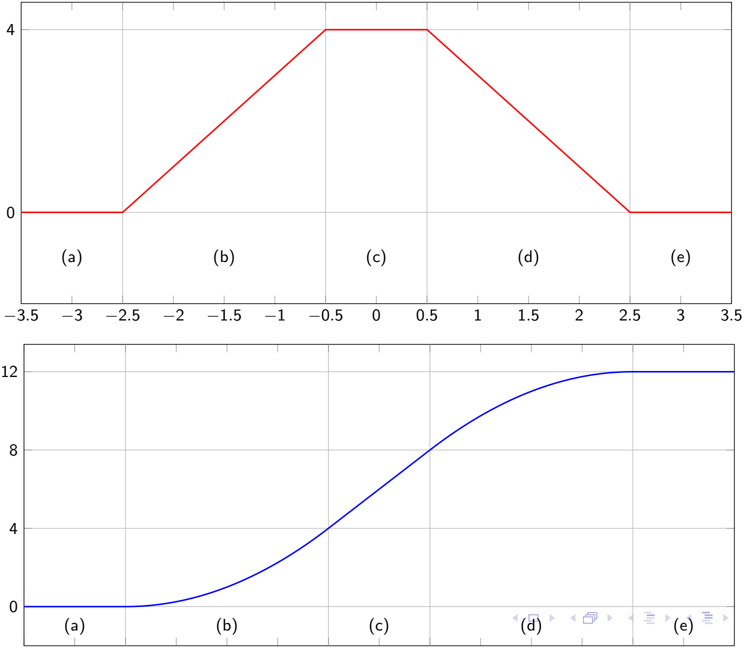
Convolution kernel computation

- The distance between the center of a pixel and the closest edge gives the coordinate along the projection (integration limits)
- Pixel area $\Leftrightarrow$ projection area (uniform unitary pixel).
- For each pixel, the overlapping area is then given by :

$$\int_0^x Trap_{p_\theta,q_\theta}(t)dt$$

- Easy to compute since we have analytic expression of the wedge

Wedge integration



Wedge integration

Closed formula to compute the area from $x_{p_\theta,q_\theta}(k,l)$:

$$K_{p_\theta,q_\theta}(x) = \begin{cases} 2|p_\theta q_\theta| - K(p_\theta, q_\theta, -x) & \text{if } x < 0 \\ |p_\theta q_\theta| + 2x\min\{|p_\theta|, |q_\theta|\} & \text{if } 0 \leqslant x < \frac{\big||p_\theta|-|q_\theta|\big|}{2} \\ |p_\theta q_\theta| - x^2 + (|p_\theta| + |q_\theta|)x & \\ -\left(\frac{|p_\theta|-|q_\theta|}{2}\right)^2 & \text{if } \frac{\big||p_\theta|-|q_\theta|\big|}{2} \leqslant x < \frac{|p_\theta|+|q_\theta|}{2} \\ 2|p_\theta q_\theta| & \text{elsewhere.} \end{cases}$$

Discrete formulation

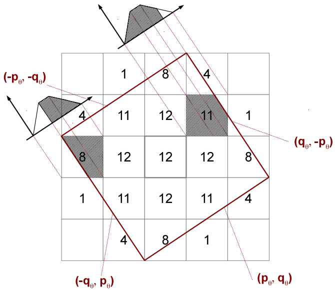
- The discrete wedge can be computed from discrete convolutions of discrete p and q wide Haar functions :

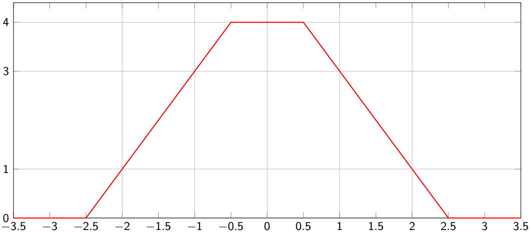
$$Trap_{p_\theta,q_\theta} = \{(1\ 1)\} * (1 \cdots 1) * (1 \cdots 1)$$

- The samples obtained are exactly those needed for the integration $\rightarrow$ fully discrete process
- Mojette projection of the convolution kernel = 1D convolution kernel to apply on projections

Discrete formulation : example

For angle $(3,2)$ rotation :





$$(1\ 1) * (1\ 1\ 1) * (1\ 1) = (1\ 3\ 4\ 3\ 1) \implies (1\ 4\ 8\ 11\ 12)$$

A few examples

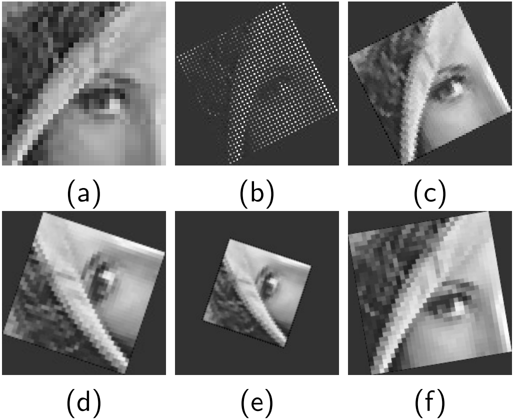


FIGURE: (a) Original image 32×32. (b,c,d,f) Mojette space rotations $(2,1)$, $(1,3)$, $(6,1)$.

Conclusion and perspectives

Conclusion

- Mojette space rotation method
  - Based on discrete geometry
  - In Mojette projection space
- Revertible rotations
- Well suited for coupled tomography and rotation problems
- Future works : 3D extension (not trivial !)

Perspectives

Other geometric transforms :

- Integer vector (d_i, d_j) translation
  - Shift of every bins :

$$\Delta b_{(p,q)}(d_i, d_j) = pd_j - qd_i$$

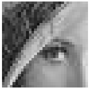
- Upscaling of factor $s \in \mathbb{N}^*$
  - Oversampling of projections :

$$b' = b \times s$$

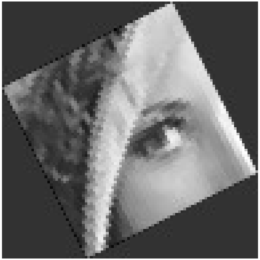
$\Rightarrow$ Definition of a similarity group in the Mojette projections space

Perspectives

Application perspective : acquisition registration



$\longrightarrow$



- Used to determine the similarity between to images, regardless of their orientation and scale.
- Used to estimate the affine transform parameters between the two images or volumes

$\Rightarrow$ Directly from the acquired Mojette projections, without the need to reconstruct the whole volume

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